Functional treatment of quantum scattering via the dynamical principle

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Abstract

A careful functional treatment of quantum scattering is given using Schwinger's dynamical principle which involves a functional differentiation operation applied to a generating functional written in closed form. For long range interactions, such as for the Coulomb one, it is shown that this expression may be used to obtain explicitly the asymptotic "free" modified Green function near the energy shell.

KEY WORDS: quantum dynamical principle, quantum scattering, long range potentials, Green functions.
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1 Introduction

The purpose of this communication is to use Schwinger's [6, 10–13] most elegant quantum dynamical principle to provide a careful functional treatment of quantum scattering. We derive rigorously an expression for the scattering amplitude involving a functional differentiation operation applied to a functional, depending on the potential, written in closed form. The main result of this paper is given in Eq.(2.28). In particular, it provides a systematic starting point for studies of deviations from so-called straight-line "trajectories" of particles, with small deviation angles, by mere functional

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differentiations. An investigation of a time limit of a function related to this expression shows that the latter may be also used to obtain the asymptotic "free" modified Green functions for theories with long range potentials such as for the Coulomb potential with the latter defining the transitional potential between short and long range potentials. Functional methods have been also introduced earlier in the literature [1–5, 9, 14–17] in quantum scattering dealing with path integrals or variational optimization methods which, however, are not in the spirit of the present paper based on the dynamical principle. The present study is an adaptation of quantum field theory methods [7] to quantum potential scattering.

2 Functional treatment of scattering

Given a Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) \tag{2.1}$$

for a particle of mass m interacting with a potential $V(\mathbf{x})$, we introduce a Hamiltonian $H'(\lambda, \tau)$ involving external sources $\mathbf{F}(\tau)$, $\mathbf{S}(\tau)$ coupled linearly to $\mathbf{x}$ and $\mathbf{p}$ as follows:

$$H'(\lambda, \tau) = \frac{\mathbf{p}^2}{2m} + \lambda V(\mathbf{x}) - \mathbf{x} \cdot \mathbf{F}(\tau) + \mathbf{p} \cdot \mathbf{S}(\tau) \tag{2.2}$$

where λ is an arbitrary parameter which will be eventually set equal to one. Schwinger's ([10]- [13], [6]) dynamical principle states, that the variation of the transformation function $\langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle$ with respect to the parameter λ for the theory governed by the Hamiltonian $H'(\lambda, \tau)$ is given by

$$\delta \langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle = \left(-\frac{\mathrm{i}}{\hbar}\right) \int_{t'}^{t} \mathrm{d}\tau \, \delta \left(\lambda V \left(-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{F}(\tau)} \right) \right) \langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle . \tag{2.3}$$

Here $V(-\mathrm{i}\hbar\delta/\delta\mathbf{F}(\tau))$ denotes $V(\mathbf{x})$ with $\mathbf{x}$ in it replaced by $-\mathrm{i}\hbar\delta/\delta\mathbf{F}(\tau)$. Eq.(2.3) may be readily integrated for $\lambda = 1$, $\mathbf{F}(\tau)$, $\mathbf{S}(\tau)$ set equal to zero, that is for the theory governed by the Hamiltonian H in (2.1), to obtain

$$\langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle = \exp \left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \, V \left(-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{F}(\tau)} \right) \right] \langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle^{(0)} \Big|_{\mathbf{F}=0, \mathbf{S}=0} \tag{2.4}$$

The transformation function $\langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle^{(0)}$ corresponds a theory developing in time via the Hamiltonian

$$H'(0,\tau) = \frac{\mathbf{p}^2}{2m} - \mathbf{x}\cdot\mathbf{F}(\tau) + \mathbf{p}\cdot\mathbf{S}(\tau), \tag{2.5}$$

to which we now pay special attention.

With $\mathbf{p}$ replaced by $\mathrm{i}\hbar\delta/\delta\mathbf{S}(\tau)$, the dynamical principle, exactly as in (2.4), gives

$$\langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle^{(0)} = \exp\left[-\frac{\mathrm{i}}{2m\hbar}\int_{t'}^{t} \mathrm{d}\tau \left(\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right)^2\right] \langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle_0 \tag{2.6}$$

where the transformation function $\langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle_0$ is governed by the "Hamiltonian"

$$\hat{H}(\tau) = -\mathbf{x}\cdot\mathbf{F}(\tau) + \mathbf{p}\cdot\mathbf{S}(\tau), \tag{2.7}$$

involving no kinetic energy term.

The Heisenberg equations corresponding to $\hat{H}(\tau)$ give the equations

$$\mathbf{x}(\tau) = \mathbf{x}(t) - \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau' - \tau)\mathbf{S}(\tau'), \tag{2.8}$$

$$\mathbf{p}(\tau) = \mathbf{p}(t') + \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau - \tau')\mathbf{F}(\tau'), \tag{2.9}$$

where $\Theta(\tau)$ is the step function $\Theta(\tau) = 1$ for $\tau > 0$ and $= 0$ for $\tau < 0$. Using the relations

$${}_0\langle \mathbf{x}t| \, \mathbf{x}(t) = \mathbf{x} \, \langle \mathbf{x}t| \,, \tag{2.10}$$

$$\mathbf{p}(t') \, |\mathbf{p}t'\rangle_0 = |\mathbf{p}t'\rangle \, \mathbf{p}, \tag{2.11}$$

and the dynamical principle, we obtain from taking the matrix elements of $\mathbf{x}(\tau)$, $\mathbf{p}(\tau)$ in (2.8),(2.9) between the states ${}_0\langle \mathbf{x}t|$, $|\mathbf{p}t'\rangle_0$, the functional differential equations

$$-\,\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)} \langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle_0 = \left[\mathbf{x} - \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau' - \tau)\mathbf{S}(\tau')\right] \langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle_0 \tag{2.12}$$

$$\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)} \langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle_0 = \left[\mathbf{p} + \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau - \tau')\mathbf{F}(\tau')\right] \langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle_0 \tag{2.13}$$

These equations may be integrated to yield

$$\langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle_0 = \exp\left[\frac{\mathrm{i}}{\hbar}\mathbf{x}\cdot\left(\mathbf{p} + \int_{t'}^{t} \mathrm{d}\tau \, \mathbf{F}(\tau)\right)\right] \exp\left[-\frac{\mathrm{i}}{\hbar}\mathbf{p}\cdot\int_{t'}^{t} \mathrm{d}\tau \, \mathbf{S}(\tau)\right]$$
$$\times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau' \, \mathbf{S}(\tau)\cdot\mathbf{F}(\tau')\Theta(\tau - \tau')\right], \tag{2.14}$$

satisfying the familiar boundary condition $\exp(\mathrm{i}\mathbf{x}\cdot\mathbf{p}/\hbar)$ for $\mathbf{F},\mathbf{S}$ set equal to zero.

Since we are interested in (2.4), in particular, for the case when $\mathbf{S}$ is set equal to zero, the functional differentiation in (2.6) may easily carried out giving

$$
\begin{aligned}
\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle^{(0)}\Big|_{\mathbf{S}=\mathbf{0}} &= \exp\left[\frac{\mathrm{i}}{\hbar}\left(\mathbf{x}\cdot\mathbf{p}-\frac{\mathbf{p}^2}{2m}(t-t')\right)\right] \\
&\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{F}(\tau)\cdot\left(\mathbf{x}-\frac{\mathbf{p}}{m}(t-\tau)\right)\right] \\
&\times\exp\left[-\frac{\mathrm{i}}{2m\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{F}(\tau)\cdot\mathbf{F}(\tau')(t-\tau_>)\right], \qquad (2.15)
\end{aligned}
$$

where $\tau_>$ is the largest of τ and $\tau' : \tau = max(\tau,\tau')$.

We recall from (2.4) that we eventually set $\mathbf{F}(\tau)$ equal to zero. This allows us to interchange the exponential factor in (2.4) involving the $V(-\mathrm{i}\hbar\delta/\delta\mathbf{F}(\tau))$ term and the last two exponential factors in (2.15). This gives for $\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle$ in (2.4) the expression

$$
\begin{aligned}
\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle &= \exp\left[\frac{\mathrm{i}}{\hbar}\left(\mathbf{x}\cdot\mathbf{p}-\frac{\mathbf{p}^2}{2m}(t-t')\right)\right] \\
&\times\exp\left[\frac{\mathrm{i}\hbar}{2m}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,[t-\tau_>]\,\frac{\delta}{\delta\mathbf{F}(\tau)}\cdot\frac{\delta}{\delta\mathbf{F}(\tau')}\right] \\
&\times\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,V\left(\mathbf{x}-\frac{\mathbf{p}}{m}(t-\tau)+\mathbf{F}(\tau)\right)\right]\Bigg|_{\mathbf{F}=\mathbf{0}}. \qquad (2.16)
\end{aligned}
$$

Since we finally set $\mathbf{F}=\mathbf{0}$ in (2.16), the theory becomes translational invariant in time and $\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle$ is a function of $t-t'\equiv T$

For $t>t'$, we have the definition of the Green function

$$
\langle \mathbf{x}t\,|\,\mathbf{x}'t'\rangle = G_+(\mathbf{x}t,\mathbf{x}'t'), \qquad (2.17)
$$

with $G_+(\mathbf{x}t,\mathbf{x}'t')$=0 for $t<t'$, and

$$
\begin{aligned}
\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle &= G_+(\mathbf{x}t,\mathbf{p}t') \\
&= \int \mathrm{d}^3\mathbf{x}'\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}'/\hbar}G_+(\mathbf{x}t,\mathbf{x}'t'). \qquad (2.18)
\end{aligned}
$$

We may now introduce the Fourier transform defined by

$$
G_+(\mathbf{p},\mathbf{p}';p^0) = -\frac{\mathrm{i}}{\hbar}\frac{1}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}(p^0+\mathrm{i}\epsilon)T/\hbar}\int\mathrm{d}^3\mathbf{x}\,\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\,\langle \mathbf{x}T\,|\,\mathbf{p}'0\rangle\,, \quad (2.19)
$$

for $\epsilon \to +0$, where $\langle \mathbf{x}T \,|\, \mathbf{p}0\rangle$ is given in (2.16) with $t - t' \equiv T$. From (2.19), (2.16), we may rewrite $G_+(\mathbf{p},\mathbf{p}';p^0)$ as

$$G_+(\mathbf{p},\mathbf{p}';p^0) = -\frac{\mathrm{i}}{\hbar}\frac{1}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}\alpha\,\mathrm{e}^{\mathrm{i}[p^0 - E(\mathbf{p}')+\mathrm{i}\epsilon]\alpha/\hbar}\int \mathrm{d}^3\mathbf{x}\,\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}K(\mathbf{x},\mathbf{p}';\alpha), \tag{2.20}$$

where $E(\mathbf{p}) = \mathbf{p}^2/2m$,

$$K(\mathbf{x},\mathbf{p}';\alpha) = \exp\left[\frac{\mathrm{i}\hbar}{2m}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'[t-\tau_>]\frac{\delta}{\delta\mathbf{F}(\tau)}\cdot\frac{\delta}{\delta\mathbf{F}(\tau')}\right] \times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\, V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}(t-\tau) + \mathbf{F}(\tau)\right)\right]\Bigg|_{\mathbf{F}=\mathbf{0}} \tag{2.21}$$

with $t - t' \equiv \alpha$ playing the role of time $-$ a notation used for it quite often in field theory.

In the α-integrand in the exponential in (2.20), we recognize $[p^0 - E(\mathbf{p}) + \mathrm{i}\epsilon]$ as the inverse of the free Green function in the energy-momentum representation.

The scattering amplitude $f(\mathbf{p},\mathbf{p}')$ for scattering of the particle with initial and final momenta $\mathbf{p}'$, $\mathbf{p}$, respectively, is defined by

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{p}''\, V(\mathbf{p}-\mathbf{p}'')G_+(\mathbf{p}'',\mathbf{p}';p^0)[p^0 - E(\mathbf{p}')]\Big|_{p^0 = E(\mathbf{p}')} \tag{2.22}$$

where $V(\mathbf{p}) = \int \mathrm{d}^3\mathbf{x}\,\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\mathbf{p}/\hbar}V(\mathbf{x})$. This suggests to multiply (2.20) by $[p^0 - E(\mathbf{p}')]$ giving

$$G_+(\mathbf{p},\mathbf{p}';p^0)[p^0 - E(\mathbf{p}')] = -\frac{1}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}\alpha\left(\frac{\partial}{\partial\alpha}\mathrm{e}^{\mathrm{i}\alpha[p^0 - E(\mathbf{p}')+\mathrm{i}\epsilon]/\hbar}\right) \times \int \mathrm{d}^3\mathbf{x}\,\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}K(\mathbf{x},\mathbf{p}';\alpha). \tag{2.23}$$

From the fact that $\langle \mathbf{x} \,|\, \mathbf{p}\rangle = \exp(\mathrm{i}\mathbf{x}\cdot\mathbf{p}/\hbar)$ and the definition of $K(\mathbf{x},\mathbf{p}';\alpha)$ in (2.21), we have

$$K(\mathbf{x},\mathbf{p}';0) = 1. \tag{2.24}$$

We now consider the cases for which

$$\lim_{\alpha\to\infty}\int \mathrm{d}^3\mathbf{x}\,\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}K(\mathbf{x},\mathbf{p}';\alpha), \tag{2.25}$$

exists. This, in particular, implies that ($\epsilon > 0$)

$$\lim_{\alpha\to\infty}\mathrm{e}^{-\epsilon\alpha}\int \mathrm{d}^3\mathbf{x}\,\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}K(\mathbf{x},\mathbf{p}';\alpha) = 0. \tag{2.26}$$

We may then integrate over α in (2.23) to obtain simply

$$\begin{aligned} & G_+(\mathbf{p},\mathbf{p}';p^0)[p^0-E(\mathbf{p}')]\Big|_{p^0=E(\mathbf{p}')} \\ & \quad = \lim_{\alpha\to\infty}\frac{1}{(2\pi\hbar)^3}\int \mathrm{d}^3\mathbf{x}\,\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}K(\mathbf{x},\mathbf{p}';\alpha), \end{aligned} \tag{2.27}$$

on the energy shell $p^0 = E(\mathbf{p})$, and for the scattering amplitude, in (2.22), *after* integrating over $\mathbf{p}''$, the expression

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\lim_{\alpha\to\infty}\int \mathrm{d}^3\mathbf{x}\,\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}V(\mathbf{x})K(\mathbf{x},\mathbf{p}';\alpha), \tag{2.28}$$

with $K(\mathbf{x},\mathbf{p}';\alpha)$ defined in (2.21). Here we recognize that the formal replacement of $K(\mathbf{x},\mathbf{p}';\alpha)$ by one gives the celebrated Born approximation. On the other hand, part of the argument $[\mathbf{x}-\mathbf{p}'(t-\tau)/m]$ of $V(\mathbf{x}-\mathbf{p}'(t-\tau)/m+\mathbf{F}(\tau))$ in (2.21), represents a "straight line trajectory" of a particle, with the functional differentiations with respect to $\mathbf{F}(\tau)$, as defined in (2.21), leading to deviations of the dynamics from such a straight line trajectory. With a straight line approximation, ignoring all of the functional differentiations, with respect to $\mathbf{F}(\tau)$ and setting the latter equal to zero, gives the following explicit expression for the scattering amplitude $f(\mathbf{p},\mathbf{p}')$ in (2.28):

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{x}\,\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}V(\mathbf{x})\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}\alpha\, V\left(\mathbf{x}-\frac{\mathbf{p}'}{m}\alpha\right)\right]. \tag{2.29}$$

This modifies the Born approximation by the presence of an additional phase factor in the integrand in (2.29), depending on the potential, accumulated during the scattering process. Here one recognizes the expression which leads to scattering with small deflections at high energies (the so-called eikonal approximation) obtained from the straight line trajectory approximation discussed above. Deviations from this approximation may be then systematically obtained by carrying out a functional power series expansion of $V(\mathbf{x}-\mathbf{p}'(t-\tau)/m+\mathbf{F}(\tau))$ in $\mathbf{F}(\tau)$ and performing the functional differential operation as dictated by the first exponential in (2.21) and finally setting $\mathbf{F}(\tau)$ equal to zero.

We note that formally that the τ-integral, involving the potential V, in (2.21) increases with no bound for $\alpha\to\infty$ for the Coulomb potential and for potentials of longer range with the former potential defining the transitional potential between long and short range potentials. And in case that the limit in (2.25) does not exist, as encountered for the Coulomb potential, (2.23) cannot be integrated by parts. This is discussed in the next section.

3 Asymptotic "free" Green function

In case the $\alpha \to \infty$ limit in (2.25) does not exist, one may study the behaviour of $G_+(\mathbf{p}, \mathbf{p}'; p^0)$ near the energy shell $p^0 \simeq \mathbf{p}'^2/2m$ directly from (2.20). To this end, we introduce the integration variable

$$z = \frac{\alpha}{\hbar}\left[p^0 - E(\mathbf{p}')\right], \tag{3.1}$$

in (2.20), to obtain

$$\begin{aligned} G_+(\mathbf{p}, \mathbf{p}'; p^0)[p^0 - E(\mathbf{p}')] = &-\frac{\mathrm{i}}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}z\, \mathrm{e}^{\mathrm{i}z(1+\mathrm{i}\epsilon)} \\ &\times \int \mathrm{d}^3\mathbf{x}\, \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}\, K\left(\mathbf{x}, \mathbf{p}'; \frac{z\hbar}{p^0 - E(\mathbf{p}')}\right), \end{aligned} \tag{3.2}$$

for $\epsilon \to +0$. For $p^0 - E(\mathbf{p}') \gtrsim 0$, i.e., near the energy shell, we may substitute

$$K\left(\mathbf{x}, \mathbf{p}'; z\hbar/(p^0 - E(\mathbf{p}'))\right) \simeq \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^{z\hbar/(p^0-E(\mathbf{p}'))} \mathrm{d}\alpha\, V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right)\right], \tag{3.3}$$

in (3.2) to obtain for the following integral

$$\begin{aligned} \int \mathrm{d}^3\mathbf{p}\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}/\hbar} G_+(\mathbf{p}, \mathbf{p}'; p^0) \simeq &\frac{-\mathrm{i}\mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\mathbf{p}'/\hbar}}{[p^0 - E(\mathbf{p}') + \mathrm{i}\epsilon]}\int_0^\infty \mathrm{d}z\, \mathrm{e}^{\mathrm{i}z(1+\mathrm{i}\epsilon)} \\ &\times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^{z\hbar/(p^0-E(\mathbf{p}'))} \mathrm{d}\alpha\, V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right)\right], \end{aligned} \tag{3.4}$$

For the Coulomb potential $V(\mathbf{x}) = \lambda/|\mathbf{x}|$,

$$\int_0^{z\hbar/(p^0-E(\mathbf{p}'))} \mathrm{d}\alpha\, V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right) \simeq \frac{\lambda m}{|\mathbf{p}'|}\ln\left(\frac{2|\mathbf{p}'|z\hbar}{m(p^0 - E(\mathbf{p}'))|\mathbf{x}|(1-\cos\theta)}\right) \tag{3.5}$$

where $\cos\theta = \mathbf{p}'\cdot\mathbf{x}/|\mathbf{p}'||\mathbf{x}|$. Hence

$$\begin{aligned} \exp -\frac{\mathrm{i}}{\hbar}\int_0^{z\hbar/(p^0-E(\mathbf{p}'))} \mathrm{d}\alpha\, V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right) \simeq &\frac{1}{[p^0 - E(\mathbf{p}') + \mathrm{i}\epsilon]^{-\mathrm{i}\gamma}} \\ &\times \exp -\mathrm{i}\gamma \ln\left(\frac{2p'^2 z\hbar}{m(p'x - \mathbf{p}'\cdot\mathbf{x})}\right), \end{aligned} \tag{3.6}$$

where $\gamma = \lambda m/\hbar p'$. Finally using the integral

$$\int_0^\infty \mathrm{d}z\, \mathrm{e}^{\mathrm{i}z(1+\mathrm{i}\epsilon)}(z)^{-\mathrm{i}\gamma} = \mathrm{i}\mathrm{e}^{\pi\gamma/2}\Gamma(1-\mathrm{i}\gamma), \tag{3.7}$$

for $\epsilon \to +0$, where Γ is the gamma function, we obtain from (3.4)

$$\int \mathrm{d}^3\mathbf{p}\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}/\hbar} G_+(\mathbf{p},\mathbf{p}';p^0) \simeq \mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\mathbf{p}'/\hbar} \frac{\mathrm{e}^{-\mathrm{i}\gamma\ln(2p'^2/m)}}{[p^0 - E(\mathbf{p}') + \mathrm{i}\epsilon]^{1-\mathrm{i}\gamma}} \times \exp \mathrm{i}\gamma \ln\left(\frac{p'x - \mathbf{p}'\cdot\mathbf{x}}{\hbar}\right) \mathrm{e}^{\pi\gamma/2}\Gamma(1-\mathrm{i}\gamma), \tag{3.8}$$

to be compared with earlier results (e.g., [8]), and for the asymptotic "free" Green function, in the energy-momentum representation, the expression.

$$G_+^0(\mathbf{p}) = \frac{\mathrm{e}^{-\mathrm{i}\gamma\ln(2p^2/m)}}{[p^0 - E(\mathbf{p}) + \mathrm{i}\epsilon]^{1-\mathrm{i}\gamma}} \mathrm{e}^{\pi\gamma/2}\Gamma(1-\mathrm{i}\gamma). \tag{3.9}$$

showing on obvious modification from the Fourier transform of the free Green function $[p^0 - E(\mathbf{p}) + \mathrm{i}\epsilon]^{-1}$.

4 Conclusion

The expression (2.28) provides a functional expression for the scattering amplitude with $K(\mathbf{x},\mathbf{p}';\alpha)$ defined in (2.21) and the latter is obtained by the functional differential operation carried out on the functional, involving the potential V, of argument $\mathbf{x}-\mathbf{p}'(t-\tau)/m+\mathbf{F}(\tau)$, for all $t' \leq \tau \leq t$, represented by the first exponential in (2.21). The "straight line trajectory" approximation of a particle consisting of retaining $\mathbf{x}-\mathbf{p}'(t-\tau)/m$ only in the argument of V and neglecting the functional differentiations with respect to $\mathbf{F}(\tau)$, with the latter operation leading systematically to modifications from this linear "trajectory", gives rise to the familiar eikonal approximation. The existence of the time limit in (2.25) distinguishes between so-called potentials of short and long ranges with the Coulomb potential providing the transitional potential between these two general classes of potentials and belonging to the latter class. In case the time limit in (2.25) does not exist, corresponding to potentials of long ranges, (2.20) may be used to obtain the asymptotic "free" modified Green function near the energy shell as seen in Sect.3. In a subsequent report, our functional expression in (2.28) for the scattering amplitude will be generalized for long range potentials as well.

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